16.78 ZTRANS: Z-transform package

This package is an implementation of the Z-transform of a sequence. This is the discrete analogue of the Laplace Transform.
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16.78.1 Z-Transform

The Z-Transform of a sequence $\{f_n\}$ is the discrete analogue of the Laplace Transform, and

$$\mathcal{Z}\{f_n\} = F(z) = \sum_{n=0}^{\infty} f_n z^{-n} \ .$$

This series converges in the region outside the circle $|z| = |z_0| = \limsup_{n\to\infty} \sqrt[n]{|f_n|}$.

SYNTAX: `ztrans(`f_n`, n, z)` where f_n is an expression, and n,z are identifiers.

16.78.2 Inverse Z-Transform

The calculation of the Laurent coefficients of a regular function results in the following inverse formula for the Z-Transform:
If $F(z)$ is a regular function in the region $|z| > \rho$ then $\exists$ a sequence $\{f_n\}$ with $\mathcal{Z}\{f_n\} = F(z)$ given by

$$f_n = \frac{1}{2\pi i} \oint F(z) z^{n-1} dz$$

SYNTAX: `invztrans(`$F(z)$`, z, n)` where $F(z)$ is an expression, and z,n are identifiers.

16.78.3 Input for the Z-Transform

This package can compute the Z-Transforms of the following list of f_n, and certain combinations thereof.

1	$e^{\alpha n}$	$\frac{1}{(n+k)}$
$\frac{1}{n!}$	$\frac{1}{(2n)!}$	$\frac{1}{(2n+1)!}$
$\frac{\sin(\beta n)}{n!}$	$\sin(\alpha n + \phi)$	$e^{\alpha n}\sin(\beta n)$

$\frac{\cos(\beta n)}{n!}$	$\cos(\alpha n + \phi)$	$e^{\alpha n}\cos(\beta n)$
$\frac{\sin(\beta(n+1))}{n+1}$	$\sinh(\alpha n + \phi)$	$\frac{\cos(\beta(n+1))}{n+1}$
$\cosh(\alpha n + \phi)$	$\binom{n+k}{m}$	

Other Combinations

Linearity	$\mathcal{Z}\{af_n + bg_n\} = a\mathcal{Z}\{f_n\} + b\mathcal{Z}\{g_n\}$
Multiplication by n	$\mathcal{Z}\{n^k \cdot f_n\} = -z\frac{d}{dz}\left(\mathcal{Z}\{n^{k-1} \cdot f_n, n, z\}\right)$
Multiplication by λ^n	$\mathcal{Z}\{\lambda^n \cdot f_n\} = F\left(\frac{z}{\lambda}\right)$
Shift Equation	$\mathcal{Z}\{f_{n+k}\} = z^k\left(F(z) - \sum_{j=0}^{k-1} f_j z^{-j}\right)$
Symbolic Sums	$\mathcal{Z}\left\{\sum_{k=0}^{n} f_k\right\} = \frac{z}{z-1} \cdot \mathcal{Z}\{f_n\}$
	$\mathcal{Z}\left\{\sum_{k=p}^{n+q} f_k\right\}$ combination of the above

where k,$\lambda \in \mathbf{N}-\{0\}$; and a,b are variables or fractions; and p,$q \in \mathbf{Z}$ or are functions of n; and α, β & ϕ are angles in radians.

16.78.4 Input for the Inverse Z-Transform

This package can compute the Inverse Z-Transforms of any rational function, whose denominator can be factored over **Q**, in addition to the following list of $F(z)$.

$\sin\left(\frac{\sin(\beta)}{z}\right)e^{\left(\frac{\cos(\beta)}{z}\right)}$	$\cos\left(\frac{\sin(\beta)}{z}\right)e^{\left(\frac{\cos(\beta)}{z}\right)}$
$\sqrt{\frac{z}{A}}\sin\left(\sqrt{\frac{z}{A}}\right)$	$\cos\left(\sqrt{\frac{z}{A}}\right)$
$\sqrt{\frac{z}{A}}\sinh\left(\sqrt{\frac{z}{A}}\right)$	$\cosh\left(\sqrt{\frac{z}{A}}\right)$

$z\ \log\left(\frac{z}{\sqrt{z^2-Az+B}}\right)$ $z\ \log\left(\frac{\sqrt{z^2+Az+B}}{z}\right)$

$\arctan\left(\frac{\sin(\beta)}{z+\cos(\beta)}\right)$

where k,$\lambda \in \mathbf{N}-\{0\}$ and A,B are fractions or variables ($B > 0$) and α,β, & ϕ are angles in radians.

16.78.5 Application of the Z-Transform

Solution of difference equations

In the same way that a Laplace Transform can be used to solve differential equations, so Z-Transforms can be used to solve difference equations.

Given a linear difference equation of k-th order

$$f_{n+k} + a_1 f_{n+k-1} + \ldots + a_k f_n = g_n \tag{16.97}$$

with initial conditions $f_0 = h_0$, $f_1 = h_1$, ..., $f_{k-1} = h_{k-1}$ (where h_j are given), it is possible to solve it in the following way. If the coefficients $a_1, \ldots, a_k$ are constants, then the Z-Transform of (16.97) can be calculated using the shift equation, and results in a solvable linear equation for $\mathcal{Z}\{f_n\}$. Application of the Inverse Z-Transform then results in the solution of (16.97).
If the coefficients $a_1, \ldots, a_k$ are polynomials in n then the Z-Transform of (16.97) constitutes a differential equation for $\mathcal{Z}\{f_n\}$. If this differential equation can be solved then the Inverse Z-Transform once again yields the solution of (16.97). Some examples of these methods of solution can be found in §16.78.6.

16.78.6 EXAMPLES

Here are some examples for the Z-Transform

```
1: ztrans((-1)^n*n^2,n,z);

    z*( - z + 1)
---------------------
  3       2
 z   + 3*z   + 3*z + 1
```

```
2: ztrans(cos(n*omega*t),n,z);

   z*(cos(omega*t) - z)
---------------------------
                     2
 2*cos(omega*t)*z - z  - 1

3: ztrans(cos(b*(n+2))/(n+2),n,z);

                                 z
z*( - cos(b) + log(------------------------------)*z)
                                          2
                    sqrt( - 2*cos(b)*z + z  + 1)

4: ztrans(n*cos(b*n)/factorial(n),n,z);

  cos(b)/z       sin(b)                     sin(b)
 e        *(cos(--------)*cos(b) - sin(--------)*sin(b))
                   z                        z
---------------------------------------------------------
                            z
5: ztrans(sum(1/factorial(k),k,0,n),n,z);

  1/z
 e   *z
--------
 z - 1

6: operator f$

7: ztrans((1+n)^2*f(n),n,z);

                        2
df(ztrans(f(n),n,z),z,2)*z  - df(ztrans(f(n),n,z),z)*z
+ ztrans(f(n),n,z)
```

Here are some examples for the Inverse Z-Transform

```
8: invztrans((z^2-2*z)/(z^2-4*z+1),z,n);

              n       n                 n
```

```
 (sqrt(3) - 2) *( - 1)  + (sqrt(3) + 2)
-----------------------------------------
                    2

9: invztrans(z/((z-a)*(z-b)),z,n);

  n    n
 a  - b
---------
  a - b

10: invztrans(z/((z-a)*(z-b)*(z-c)),z,n);

  n      n      n      n      n      n
 a *b - a *c - b *a + b *c + c *a - c *b
-----------------------------------------
  2      2         2      2    2        2
 a *b - a *c - a*b  + a*c  + b *c - b*c

11: invztrans(z*log(z/(z-a)),z,n);

  n
 a *a
-------
 n + 1

12: invztrans(e^(1/(a*z)),z,n);

        1
-----------------
  n
 a *factorial(n)

13: invztrans(z*(z-cosh(a))/(z^2-2*z*cosh(a)+1),z,n);

cosh(a*n)
```

Examples: Solutions of Difference Equations

I (See [1], p. 651, Example 1).
Consider the homogeneous linear difference equation

$$f_{n+5} - 2f_{n+3} + 2f_{n+2} - 3f_{n+1} + 2f_n = 0$$

with initial conditions $f_0 = 0$, $f_1 = 0$, $f_2 = 9$, $f_3 = -2$, $f_4 = 23$. The Z-Transform of the left hand side can be written as $F(z) = P(z)/Q(z)$ where $P(z) = 9z^3 - 2z^2 + 5z$ and $Q(z) = z^5 - 2z^3 + 2z^2 - 3z + 2 = (z-1)^2(z+2)(z^2+1)$, which can be inverted to give

$$f_n = 2n + (-2)^n - \cos\frac{\pi}{2}n\ .$$

The following REDUCE session shows how the present package can be used to solve the above problem.

```
14: operator f$ f(0):=0$ f(1):=0$ f(2):=9$ f(3):=-2$ f(4):=23$


20: equation:=ztrans(f(n+5)-2*f(n+3)+2*f(n+2)-3*f(n+1)+2*f(n),n

                                    5                       3
equation := ztrans(f(n),n,z)*z   - 2*ztrans(f(n),n,z)*z

                                         2
             + 2*ztrans(f(n),n,z)*z   - 3*ztrans(f(n),n,z)*z

                                             3      2
             + 2*ztrans(f(n),n,z) - 9*z   + 2*z   - 5*z


21: ztransresult:=solve(equation,ztrans(f(n),n,z));

                                                      2
                                               z*(9*z   - 2*z + 5)
ztransresult := {ztrans(f(n),n,z)=-----------------------------}
                                            5      3      2
                                           z   - 2*z   + 2*z   - 3*z + 2

22: result:=invztrans(part(first(ztransresult),2),z,n);

                   n     n        n     n
           2*( - 2)   - i *( - 1)   - i   + 4*n
result := -----------------------------------
```

2

II (See [1], p. 651, Example 2).
Consider the inhomogeneous difference equation:

$$f_{n+2} - 4f_{n+1} + 3f_n = 1$$

with initial conditions $f_0 = 0$, $f_1 = 1$. Giving

$$\begin{aligned} F(z) &= \mathcal{Z}\{1\} \left(\tfrac{1}{z^2-4z+3} + \tfrac{z}{z^2-4z+3}\right) \\ &= \tfrac{z}{z-1} \left(\tfrac{1}{z^2-4z+3} + \tfrac{z}{z^2-4z+3}\right). \end{aligned}$$

The Inverse Z-Transform results in the solution

$$f_n = \tfrac{1}{2}\left(\tfrac{3^{n+1}-1}{2} - (n+1)\right).$$

The following REDUCE session shows how the present package can be used to solve the above problem.

```
23: clear(f)$ operator f$ f(0):=0$ f(1):=1$


27: equation:=ztrans(f(n+2)-4*f(n+1)+3*f(n)-1,n,z);

                                    3                      2
equation := (ztrans(f(n),n,z)*z  - 5*ztrans(f(n),n,z)*z

                                                             2
    + 7*ztrans(f(n),n,z)*z - 3*ztrans(f(n),n,z) - z )/(z - 1)

28: ztransresult:=solve(equation,ztrans(f(n),n,z));

                                                 2
                                                z
result := {ztrans(f(n),n,z)=---------------------}
                                  3       2
                                 z  - 5*z  + 7*z - 3
```

```
29: result:=invztrans(part(first(ztransresult),2),z,n);

                 n
              3*3   - 2*n - 3
result := ----------------
                    4
```

III Consider the following difference equation, which has a differential equation for $\mathcal{Z}\{f_n\}$.

$$(n+1) \cdot f_{n+1} - f_n = 0$$

with initial conditions $f_0 = 1$, $f_1 = 1$. It can be solved in REDUCE using the present package in the following way.

```
30: clear(f)$ operator f$ f(0):=1$ f(1):=1$


34: equation:=ztrans((n+1)*f(n+1)-f(n),n,z);

                                              2
equation :=  - (df(ztrans(f(n),n,z),z)*z   + ztrans(f(n),n,z))

35: operator tmp;

36: equation:=sub(ztrans(f(n),n,z)=tmp(z),equation);

                                    2
equation :=  - (df(tmp(z),z)*z   + tmp(z))

37: load(odesolve);

38: ztransresult:=odesolve(equation,tmp(z),z);

                          1/z
ztransresult := {tmp(z)=e    *arbconst(1)}

39: preresult:=invztrans(part(first(ztransresult),2),z,n);
```

```
                arbconst(1)
preresult := --------------
                factorial(n)

40: solve({sub(n=0,preresult)=f(0),sub(n=1,preresult)=f(1)},
arbconst(1));

{arbconst(1)=1}

41: result:=preresult where ws;

                  1
result := --------------
             factorial(n)
```

Bibliography

[1] Bronstein, I.N. and Semedjajew, K.A., *Taschenbuch der Mathematik*, Verlag Harri Deutsch, Thun und Frankfurt(Main), 1981. ISBN 3 87144 492 8.