16.35 LIE: Functions for the classification of real n-dimensional Lie algebras

LIE is a package of functions for the classification of real n-dimensional Lie algebras. It consists of two modules: **liendmc1** and **lie1234**. With the help of the functions in the **liendmcl** module, real n-dimensional Lie algebras L with a derived algebra $L^{(1)}$ of dimension 1 can be classified.

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liendmc1

With the help of the functions in this module real n-dimensional Lie algebras L with a derived algebra $L^{(1)}$ of dimension 1 can be classified. L has to be defined by its structure constants c_{ij}^k in the basis $\{X_1, \ldots, X_n\}$ with $[X_i, X_j] = c_{ij}^k X_k$. The user must define an ARRAY LIENSTRUCIN(n, n, n) with n being the dimension of the Lie algebra L. The structure constants LIENSTRUCIN(i, j, k):=c_{ij}^k for $i < j$ should be given. Then the procedure LIENDIMCOM1 can be called. Its syntax is:

```
LIENDIMCOM1(<number>).
```

`<number>` corresponds to the dimension n. The procedure simplifies the structure of L performing real linear transformations. The returned value is a list of the form

```
(i) {LIE_ALGEBRA(2),COMMUTATIVE(n-2)} or
(ii) {HEISENBERG(k),COMMUTATIVE(n-k)}
```

with $3 \leq k \leq n$, k odd.
The concepts correspond to the following theorem (`LIE_ALGEBRA(2)` $\rightarrow L_2$, `HEISENBERG(k)` $\rightarrow H_k$ and `COMMUTATIVE(n-k)` $\rightarrow C_{n-k}$):

Theorem. Every real n-dimensional Lie algebra L with a 1-dimensional derived algebra can be decomposed into one of the following forms:

(i) $C(L) \cap L^{(1)} = \{0\} \,:\, L_2 \oplus C_{n-2}$ or
(ii) $C(L) \cap L^{(1)} = L^{(1)} \,:\, H_k \oplus C_{n-k} \quad (k = 2r - 1,\, r \geq 2)$, with

1. $C(L) = C_j \oplus (L^{(1)} \cap C(L))$ and $\dim C_j = j$,
2. L_2 is generated by Y_1, Y_2 with $[Y_1, Y_2] = Y_1$,
3. H_k is generated by $\{Y_1, \ldots, Y_k\}$ with
   $[Y_2, Y_3] = \cdots = [Y_{k-1}, Y_k] = Y_1$.

(cf. [2])

The returned list is also stored as LIE_LIST. The matrix LIENTRANS gives the transformation from the given basis $\{X_1, \ldots, X_n\}$ into the standard basis $\{Y_1, \ldots, Y_n\}$: $Y_j = (\text{LIENTRANS})_j^k X_k$.
A more detailed output can be obtained by turning on the switch `TR_LIE`:

```
ON TR_LIE;
```

before the procedure LIENDIMCOM1 is called.

The returned list could be an input for a data bank in which mathematical relevant properties of the obtained Lie algebras are stored.

lie1234

This part of the package classifies real low-dimensional Lie algebras L of the dimension $n :=\dim L = 1, 2, 3, 4$. L is also given by its structure constants c_{ij}^k in the basis $\{X_1, \ldots, X_n\}$ with $[X_i, X_j] = c_{ij}^k X_k$. An ARRAY LIESTRIN$(n, n, n)$ has to be defined and LIESTRIN(i, j, k):=c_{ij}^k for $i < j$ should be given. Then the procedure LIECLASS can be performed whose syntax is:

```
LIECLASS(<number>).
```

`<number>` should be the dimension of the Lie algebra L. The procedure stepwise simplifies the commutator relations of L using properties of invariance like the dimension of the centre, of the derived algebra, unimodularity etc. The returned value has the form:

```
{LIEALG(n),COMTAB(m)},
```

where m corresponds to the number of the standard form (basis: $\{Y_1, \ldots, Y_n\}$) in an enumeration scheme. The corresponding enumeration schemes are listed below (cf. [3],[1]). In case that the standard form in the enumeration scheme depends on one (or two) parameter(s) p_1 (and p_2) the list is expanded to:

```
{LIEALG(n),COMTAB(m),p1,p2}.
```

This returned value is also stored as LIE_CLASS. The linear transformation from the basis $\{X_1, \ldots, X_n\}$ into the basis of the standard form $\{Y_1, \ldots, Y_n\}$ is given by the matrix LIEMAT: $Y_j = (\text{LIEMAT})_j^k X_k$.

By turning on the switch TR_LIE:

```
ON TR_LIE;
```

before the procedure LIECLASS is called the output contains not only the list LIE_CLASS but also the non-vanishing commutator relations in the standard form.

By the value m and the parameters further examinations of the Lie algebra are possible, especially if in a data bank mathematical relevant properties of the enumerated standard forms are stored.

Enumeration schemes for lie1234

returned list LIE_CLASS	the corresponding commutator relations
LIEALG(1),COMTAB(0)	commutative case
LIEALG(2),COMTAB(0)	commutative case
LIEALG(2),COMTAB(1)	$[Y_1, Y_2] = Y_2$
LIEALG(3),COMTAB(0)	commutative case
LIEALG(3),COMTAB(1)	$[Y_1, Y_2] = Y_3$
LIEALG(3),COMTAB(2)	$[Y_1, Y_3] = Y_3$
LIEALG(3),COMTAB(3)	$[Y_1, Y_3] = Y_1, [Y_2, Y_3] = Y_2$
LIEALG(3),COMTAB(4)	$[Y_1, Y_3] = Y_2, [Y_2, Y_3] = Y_1$
LIEALG(3),COMTAB(5)	$[Y_1, Y_3] = -Y_2, [Y_2, Y_3] = Y_1$
LIEALG(3),COMTAB(6)	$[Y_1, Y_3] = -Y_1 + p_1 Y_2, [Y_2, Y_3] = Y_1, p_1 \neq 0$
LIEALG(3),COMTAB(7)	$[Y_1, Y_2] = Y_3, [Y_1, Y_3] = -Y_2, [Y_2, Y_3] = Y_1$
LIEALG(3),COMTAB(8)	$[Y_1, Y_2] = Y_3, [Y_1, Y_3] = Y_2, [Y_2, Y_3] = Y_1$
LIEALG(4),COMTAB(0)	commutative case
LIEALG(4),COMTAB(1)	$[Y_1, Y_4] = Y_1$
LIEALG(4),COMTAB(2)	$[Y_2, Y_4] = Y_1$
LIEALG(4),COMTAB(3)	$[Y_1, Y_3] = Y_1, [Y_2, Y_4] = Y_2$
LIEALG(4),COMTAB(4)	$[Y_1, Y_3] = -Y_2, [Y_2, Y_4] = Y_2,$ $[Y_1, Y_4] = [Y_2, Y_3] = Y_1$
LIEALG(4),COMTAB(5)	$[Y_2, Y_4] = Y_2, [Y_1, Y_4] = [Y_2, Y_3] = Y_1$
LIEALG(4),COMTAB(6)	$[Y_2, Y_4] = Y_1, [Y_3, Y_4] = Y_2$
LIEALG(4),COMTAB(7)	$[Y_2, Y_4] = Y_2, [Y_3, Y_4] = Y_1$
LIEALG(4),COMTAB(8)	$[Y_1, Y_4] = -Y_2, [Y_2, Y_4] = Y_1$
LIEALG(4),COMTAB(9)	$[Y_1, Y_4] = -Y_1 + p_1 Y_2, [Y_2, Y_4] = Y_1, p_1 \neq 0$
LIEALG(4),COMTAB(10)	$[Y_1, Y_4] = Y_1, [Y_2, Y_4] = Y_2$
LIEALG(4),COMTAB(11)	$[Y_1, Y_4] = Y_2, [Y_2, Y_4] = Y_1$

returned list LIE_CLASS	the corresponding commutator relations
LIEALG(4),COMTAB(12)	$[Y_1, Y_4] = Y_1 + Y_2, [Y_2, Y_4] = Y_2 + Y_3,$ $[Y_3, Y_4] = Y_3$
LIEALG(4),COMTAB(13)	$[Y_1, Y_4] = Y_1, [Y_2, Y_4] = p_1 Y_2, [Y_3, Y_4] = p_2 Y_3,$ $p_1, p_2 \neq 0$
LIEALG(4),COMTAB(14)	$[Y_1, Y_4] = p_1 Y_1 + Y_2, [Y_2, Y_4] = -Y_1 + p_1 Y_2,$ $[Y_3, Y_4] = p_2 Y_3, p_2 \neq 0$
LIEALG(4),COMTAB(15)	$[Y_1, Y_4] = p_1 Y_1 + Y_2, [Y_2, Y_4] = p_1 Y_2,$ $[Y_3, Y_4] = Y_3, p_1 \neq 0$
LIEALG(4),COMTAB(16)	$[Y_1, Y_4] = 2Y_1, [Y_2, Y_3] = Y_1,$ $[Y_2, Y_4] = (1 + p_1)Y_2, [Y_3, Y_4] = (1 - p_1)Y_3,$ $p_1 \geq 0$
LIEALG(4),COMTAB(17)	$[Y_1, Y_4] = 2Y_1, [Y_2, Y_3] = Y_1,$ $[Y_2, Y_4] = Y_2 - p_1 Y_3, [Y_3, Y_4] = p_1 Y_2 + Y_3,$ $p_1 \neq 0$
LIEALG(4),COMTAB(18)	$[Y_1, Y_4] = 2Y_1, [Y_2, Y_3] = Y_1,$ $[Y_2, Y_4] = Y_2 + Y_3, [Y_3, Y_4] = Y_3$
LIEALG(4),COMTAB(19)	$[Y_2, Y_3] = Y_1, [Y_2, Y_4] = Y_3, [Y_3, Y_4] = Y_2$
LIEALG(4),COMTAB(20)	$[Y_2, Y_3] = Y_1, [Y_2, Y_4] = -Y_3, [Y_3, Y_4] = Y_2$
LIEALG(4),COMTAB(21)	$[Y_1, Y_2] = Y_3, [Y_1, Y_3] = -Y_2, [Y_2, Y_3] = Y_1$
LIEALG(4),COMTAB(22)	$[Y_1, Y_2] = Y_3, [Y_1, Y_3] = Y_2, [Y_2, Y_3] = Y_1$

Bibliography

[1] M.A.H. MacCallum. On the classification of the real four-dimensional lie algebras. 1979.

[2] C. Schoebel. Classification of real n-dimensional lie algebras with a low-dimensional derived algebra. In *Proc. Symposium on Mathematical Physics '92*, 1993.

[3] F. Schoebel. The symbolic classification of real four-dimensional lie algebras. 1992.